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LETTER TO THE EDITOR

On the issue of imposing boundary conditions on quantum fields

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As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality.

A Einstein

Abstract

An interesting example of the deep interrelation between physics and mathematics is obtained when trying to impose mathematical boundary conditions on physical quantum fields. This procedure has recently been re-examined with care. Comments on that and previous analysis are provided here, together with considerations on the results of the purely mathematical zeta-function method, in an attempt at clarifying the issue. Hadamard regularization is invoked in order to fill the gap between the infinities appearing in the QFT renormalized results and the finite values obtained in the literature with other procedures.

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1. Introduction

The question, phrased by Eugene Wigner as that of *the unreasonable effectiveness of mathematics in the natural sciences* [1] is an old and intriguing one. It goes back to Pythagoras and his school (*all things are numbers*), even probably to the Sumerians, and maybe to more ancient cultures, which left no trace. I Kant and A Einstein also contributed to this idea with profound reflections, and *mathematical simplicity, and beauty,* have remained for many years crucial ingredients when having to choose among different plausible possibilities.

An example of *unreasonable effectiveness* is provided by the regularization procedures in quantum field theory (QFT) based upon analytic continuation in the complex plane

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(dimensional, heat-kernel, zeta-function regularization and the like). That one obtains a physical, experimentally measurable, and extremely precise result after these weird mathematical manipulations is, if not unreasonable, certainly very mysterious. For many highly honourable physicists these remained always illegal practices. Such methods are now fully justified and blessed with Nobel Prizes, but more because of the many and very precise experimental checkouts (the *effectiveness*) than for their intrinsic *reasonableness*.

A simple example may be clarifying. Consider the calculation of the zero-point energy (vacuum to vacuum transition, also called Casimir energy [2]) corresponding to a quantum operator, H, with eigenvalues λ_n: $E_0 = \langle 0|H|0\rangle = \frac{1}{2}\sum_n \lambda_n$, where the sum over n may involve several continuum and discrete indices. Only in special cases will this sum be convergent. Generically one has a divergent series, to be regularized by different means. The zeta-function method [3]—which stands on solid and flourishing mathematical grounds [4]—will interpret it as the value of the zeta function of H: $\zeta_H(s) = \sum_n \lambda_n^{-s}$, at $s = -1$ (we set $\hbar = c = 1$). Generically $\zeta_H(s)$ is only defined as an absolutely convergent series for $\mathrm{Re}\, s > a_0$ (a_0 an abscissa of convergence), but it can be continued to the whole complex plane, with the possible appearance of poles as only singularities. If $\zeta_H(s)$ has no pole at $s = -1$ then we are done; if it hits a pole, further elaboration is necessary. That the mathematical result one thus gets coincides with the experimental one, constitutes here our specific example of *unreasonable effectiveness of mathematics*.

In fact things do not turn out to be so simple. One cannot assign a meaning to the *absolute* value of the zero-point energy, and any physical effect is an energy difference between two situations, such as a quantum field in curved space as compared with the same field in flat space, or one satisfying boundary conditions (BCs) on some surface as compared with the same in its absence, etc. This difference is the Casimir energy: $E_C = E_0^{\mathrm{BC}} - E_0 = \frac{1}{2}(\mathrm{Tr}\, H^{\mathrm{BC}} - \mathrm{Tr}\, H)$.

And here the problem appears. Imposing mathematical boundary conditions on physical quantum fields turns out to be a highly non-trivial act. This was discussed in much detail in a paper by Deutsch and Candelas a quarter of a century ago [5]. These authors quantized em and scalar fields in the region near an arbitrary smooth boundary, and calculated the renormalized vacuum expectation value of the stress–energy tensor, to find that the energy density diverges as the boundary is approached. Therefore, regularization and renormalization did not seem to cure the problem with infinities in this case and an infinite *physical* energy was obtained if the mathematical BCs were to be fulfilled. However, the authors argued that surfaces have non-zero depth, and its value could be taken as a handy (dimensional) cut-off in order to regularize the infinities. This approach will be recovered later in this paper. Just two years after Deutsch and Candelas' work, Kurt Symanzik carried out a rigorous analysis of QFT in the presence of boundaries [6]. Prescribing the value of the quantum field on a boundary means using the Schrödinger representation, and Symanzik was able to show rigorously that such representation exists to all orders in the perturbative expansion. He showed also that the field operator being diagonalized in a smooth hypersurface differs from the usual renormalized one by a factor that diverges logarithmically when the distance to the hypersurface goes to zero. This requires a precise limiting procedure and point splitting to be applied. In any case, the issue was proved to be perfectly meaningful within the domains of renormalized QFT. In this case the BCs and the hypersurfaces themselves were treated at a pure mathematical level (zero depth) by using delta functions.

Recently, a new approach to the problem has been postulated [7]. BCs on a field, ϕ, are enforced on a surface, S, by introducing a scalar potential, σ, of Gaussian shape living on and near the surface. When the Gaussian becomes a delta function, the BCs (Dirichlet here) are enforced: the delta-shaped potential destroys *all* the modes of ϕ at the surface. For the rest, the quantum system undergoes a full-fledged QFT renormalization, as in the case of Symanzik's approach. The results obtained confirm those of [5] in the several models studied albeit they do not seem to agree with those of [6]. They are also in clear contradiction to those quoted in the usual textbooks and review articles dealing with the Casimir effect [8], where no infinite energy density when approaching the Casimir plates has been reported.

2. A zeta-function approach

Too often has it been argued that sophisticated regularization methods, as the zeta-function procedure, get rid of infinities in an obscure way (e.g. through analytic continuation), so that, in contrast to what happens with cut-offs, one cannot keep trace of the infinites, which are cleared up without control, leading sometimes to erroneous results.

One cannot refute a statement of this kind rigorously, but it should be noted that more than once (if not always) the discrepancies between the result obtained by using the zeta procedure and other—say cut-off like—approaches have been proved to emerge from a *misuse* of zeta regularization, and *not* to stem from the method itself. When employed properly, the correct results have been recovered (for a good number of examples, see [3, 4, 9, 10]).

Take the most simple case of a scalar field in one dimension, $\phi(x)$, with a BC of Dirichlet type imposed at a point, e.g. $\phi(0) = 0$. We would like to calculate the Casimir energy for this configuration, that is, the difference between the zero-point energy corresponding to this field when the BC is enforced, and the zero-point energy in the absence of any BC. Taken at face value, both energies are infinite. The regularized difference may still be infinite when the BC point is approached (this is the result in [7]) or might turn out to be finite (even zero, which is the result given in some standard books on the subject).

Let us try to understand this discrepancy. We have to add up all energy modes (trace of H). For the mode with energy ω, the field equation reduces to

$$-\phi''(x) + m^2\phi(x) = \omega^2\phi(x). \qquad (1)$$

In the absence of a BC, the solutions to the field equation can be labelled by $k = +\sqrt{\omega^2 - m^2} > 0$, as $\phi_k(x) = A\,\mathrm{e}^{\mathrm{i}kx} + B\,\mathrm{e}^{-\mathrm{i}kx}$, with A, B being arbitrary complex (for the general complex), or as $\phi_k(x) = a\sin(kx) + b\cos(kx)$, with a, b being arbitrary real (for the general real solution). Now, when the mathematical BC of Dirichlet type, $\phi(0) = 0$, is imposed, this does not influence at all the eigenvalues, k, which remain exactly the same (as stressed in the literature). However, the *number* of solutions corresponding to each eigenvalue is reduced by one-half to: $\phi_k^{(D)}(x) = A(\mathrm{e}^{\mathrm{i}kx} - \mathrm{e}^{-\mathrm{i}kx})$, with A being arbitrary complex (complex solution), and $\phi_k^{(D)}(x) = a\sin(kx)$, with a being arbitray real (real solution). In other words, the energy spectrum (for omega) that we obtain in both cases is the same, a continuous spectrum $\omega = \sqrt{m^2 + k^2}$, but the number of eigenstates corresponding to a given eigenvalue is twice as large in the absence of the BC.[2]

Of course these considerations are elementary, but they seem to have been put aside sometimes. They are crucial when trying to calculate (or just to give sense to) the Casimir energy density and force. More to this, just in the same way as the traces of the two matrices $M_1 = \mathrm{diag}\ (\alpha, \beta)$ and $M_2 = \mathrm{diag}\ (\alpha, \alpha, \beta, \beta)$ are not equal in spite of having 'the same

[2] To understand this point even better (by taking recourse to what is learned in the maths classes at high school), consider the fact that further, by imposing Cauchy BC: $\phi(0) = 0, \phi'(0) = 0$, the eigenvalues still remain the *same*, but for any k the family of eigenfunctions shrinks to just the trivial one: $\phi_k(x) = 0, \forall k$ (the Cauchy problem is an initial value problem, which completely determines the solution).

spectrum α, β', in the problem under discussion the traces of the Hamiltonian with and without the Dirichlet BC imposed yield different results, both of them divergent, namely

$$\operatorname{Tr} H = 2 \operatorname{Tr} H^{\mathrm{BC}} = 2 \int_0^\infty \mathrm{d}k \sqrt{m^2 + k^2}. \qquad (2)$$

By using the zeta function, we define

$$\zeta^{\mathrm{BC}}(s) := \int_0^\infty \mathrm{d}\kappa\, (\nu^2 + \kappa^2)^{-s} \qquad \nu := \frac{m}{\mu} \qquad (3)$$

with μ being a regularization parameter with dimensions of mass[3]. We get

$$\zeta^{\mathrm{BC}}(s) = \frac{\sqrt{\pi}\,\Gamma(s - 1/2)}{2\Gamma(s)} (\nu^2)^{1/2-s} \qquad (4)$$

and consequently,

$$\begin{aligned} \operatorname{Tr} H^{\mathrm{BC}} &= \frac{1}{2}\zeta_{\mathrm{BC}}(s = -1/2) \\ &= \frac{m^2}{4\sqrt{\pi}} \left[\frac{1}{s + 1/2} + 1 - \gamma - \log \frac{m^2}{\mu^2} - \Psi(-1/2) + \mathcal{O}(s + 1/2) \right] \Bigg|_{s=-1/2}. \end{aligned} \qquad (5)$$

As is obvious, this divergence is *not cured* when taking the difference of the two traces in order to obtain the Casimir energy:

$$E_C/\mu = E_0^{\mathrm{BC}}/\mu - E_0/\mu = -E_0^{\mathrm{BC}}/\mu = \frac{\Gamma(-1)m^2}{8\mu^2}. \qquad (6)$$

We just hit the pole of the zeta function, in this case.

How is this infinite to be interpreted? What is its origin? Just by taking recourse to the pure mathematical theory (*durch reine Mathematik*), we already get a perfect description of what happens and understand well where does this infinite energy[4] come from. It clearly originates from the fact that imposing the BC has drastically reduced to one-half the family of eigenfunctions corresponding to any of the eigenvalues which constitute the spectrum of the operator. And we can also advance that, since this dramatic reduction of the family of eigenfunctions takes place precisely at the point where the BC is imposed, the physical divergence (infinite energy) will originate right there, and nowhere else.

While the analysis above cannot be taken as a substitute for the actual modelization of Jaffe *et al* [7]—where the BC is explicitly enforced through the introduction of an auxiliary, localized field, which probes what happens at the boundary in a much more precise way—it certainly shows that *pure mathematical considerations*, which include the use of analytic continuation by means of the zeta function, are in no way blind to the infinities of the physical model and *do not* produce misleading results, when the mathematics is used properly. And it is very remarkable to realize how close the mathematical description of the appearance of an infinite contribution is to that provided by the more physical realization in [7].

[3] Always necessary in zeta regularization, since the complex powers of the spectrum of a (pseudo-) differential operator can only be defined, physically, if the operator is rendered dimensionless, which is done by introducing this parameter. This is also an important issue, which is sometimes overlooked.
[4] In mathematical terms, this is the infinite value for the trace of the Hamiltonian operator.

3. The case of two-point Dirichlet boundary conditions

A similar analysis can be done for the case of a two-point Dirichlet BC: $\phi(a) = 0, \phi(-a) = 0$. Straightforward algebra shows, in this situation, that the eigenvalues k are quantized, as $k = \dot{\pi}/(2a)$, so that

$$\omega_\ell = \sqrt{m^2 + \frac{\ell^2\pi^2}{4a^2}} \qquad \ell = 0, 1, 2, \ldots \tag{7}$$

The family of eigenfunctions corresponding to a given eigenvalue, ω_ℓ, is of continuous dimension 1, exactly as in the former case of a one-point Dirichlet BC, namely, $\phi_\ell(x) = b \sin\left(\frac{\ell\pi}{2a}(x - a)\right)$, where b is an arbitrary, real parameter[5]. To repeat, the act of imposing Dirichlet BC on two points has the effect of discretizing the spectrum but there is no further shrinking in the number of eigenfunctions corresponding to a given (discrete) eigenvalue.

The calculation of the Casimir energy, by means of the zeta function, proceeds in this case as follows [3, 4, 9, 10]. To begin with, it may be interesting to recall that the zeta-'measure' of the continuum equals twice the zeta-'measure' of the discrete. In fact, just consider the following regularizations: $\sum_{n=1}^\infty \mu = \mu \sum_{n=1}^\infty n^{-s}\big|_{s=0} = \mu\zeta_R(0) = -\frac{\mu}{2}$, and $\int_\mu^\infty \mathrm{d}k = \int_0^\infty \mathrm{d}k(k+\mu)^{-s}\big|_{s=0} = \frac{\mu^{1-s}}{s-1}\big|_{s=0} = -\mu$, which prove the statement.

The trace of the Hamiltonian corresponding to the quantum system with the BC imposed, in the massive case, is obtained by means of the zeta function

$$\zeta^{\mathrm{BC}}(s) := \sum_{\ell=1}^{\infty} \left(\frac{m^2}{\mu^2} + \frac{\pi^2\ell^2}{4\mu^2 a^2}\right)^{-s}$$
$$= \left(\frac{\mu}{m}\right)^{2s} \left[-\frac{1}{2} + \frac{\Gamma(s-1/2)}{\Gamma(s)} \frac{am}{\sqrt{\pi}} + \frac{2\pi^s}{\Gamma(s)} \left(\frac{2am}{\pi}\right)^{1/2+s} \sum_{n=1}^{\infty} n^{s-1/2} K_{s-1/2}(4anm)\right] \tag{8}$$

K_ν being a modified Bessel function of the third kind (or MacDonald's function). Thus, for the zero-point energy of the system with two-point Dirichlet BC, we get

$$\mathrm{Tr}\, H^{\mathrm{BC}}/\mu = \frac{1}{2}\zeta_{\mathrm{BC}}(s = -1/2) = -\frac{\Gamma(-1)m^2}{8\mu^2} - \frac{m}{2\pi\mu} \sum_{n=1}^{\infty} \frac{1}{n} K_1(2\pi nm/\mu) \tag{9}$$

where μ is, in this case, $\mu := \pi/(2a)$ (a fixes the mass scale in a natural way here). As in the previous example, we finally obtain an infinite value for the Casimir energy, namely

$$E_C/\mu = E_0^{\mathrm{BC}}/\mu - E_0/\mu = \frac{\Gamma(-1)m^2}{8\mu^2} - \frac{m}{2\pi\mu} \sum_{n=1}^{\infty} \frac{1}{n} K_1(2\pi nm/\mu). \tag{10}$$

It is, therefore, not true that regularization methods using analytical continuation (in particular, the zeta approach) are unable to see the infinite energy that is generated on the boundary-condition surface [5–7] (see equation (19) below). The reason is still the same as in the previous example: imposing a two-point Dirichlet BC amounts again to halving the family of eigenfunctions which correspond to any given eigenvalue (all are discrete, in the present case, but this makes no difference). In physical terms, this means having to apply an infinite amount of energy on the BC sites, in order to enforce the BC. In absolute analogy, from the mathematical viewpoint, halving the family of eigenfunctions immediately results in the appearance of an infinite contribution, under the form of a pole of the zeta function.

[5] The contribution of the zero mode ($\ell = 0$) is controverted, but we are not going to discuss this issue here (see e.g. [11] and references therein).

The reason why these infinities (the one here and that in the previous section) do not usually show up in the literature on the Casimir effect is probably because textbooks on the subject focus towards the calculation of the Casimir *force*, which is obtained by taking minus the derivative of the energy with respect to the plate (or point) separation (here w.r.t. $2a$). Since the infinite terms do not depend on a, they do not contribute to the force (as is recognized explicitly in [7]). However, some erroneous statements have indeed appeared in the above-mentioned classical references, stemming from the lack of recognition of the catastrophical implications of the act of halving the number of eigenfunctions, when imposing the BC. The persistence of the eigenvalues of the spectrum was probably misleading. We hope to have clarified this issue here.

4. How to deal with the infinities

Here, the infinite contributions have shown up at the regularization level, but a more careful study [7] is able to prove that they do not disappear even after renormalizing in a proper way. The important question is now: are these infinities *physical*? Will they be *observed* as a manifestation of a very large energy pressure when approaching the BC surface in a lab experiment? No doubt such questions will be best answered in that way, e.g. experimentally. If, in contrast, this sort of large pressures fails to manifest itself, this might be a clear indication of the need for an additional regularization prescription. In principle, this seems to be forbidden by standard renormalization theory, since the procedure has been already carried out to the very end: there remains no additional physical quantity which could possibly absorb the divergences (see [7]).

In any case, there are circumstances—both in physics and in mathematics—where certain 'non-orthodox' regularization methods have been employed with promising success. In particular, Hadamard regularization in higher-post-Newtonian general relativity [12] and also in recent variants of axiomatic and constructive QFT [13]. Among mathematicians, Hadamard regularization is nowadays a rather standard technique in order to deal with singular differential and integral equations with BCs, both analytically and numerically (for a sample of references see [14]). Indeed, Hadamard regularization is a well-established procedure in order to give sense to infinite integrals. It is not to be found in the classical books on infinite calculus by Hardy or Knopp; it was Schwartz [15] who popularized it, rescuing Hadamard's original papers. Nowadays, Hadamard convergence is one of the cornerstones in the rigorous formulation of QFT through micro-localization, which on its turn is considered by specialists to be the most important step towards the understanding of linear PDEs since the invention of distributions (for a beautiful, updated treatment of Hadamard's regularization see [16]).

Let us briefly recall this formulation. Consider a function, $g(x)$, expandable as

$$g(x) = \sum_{j=1}^{k} \frac{a_j}{(x-a)^{\lambda_j}} + h(x) \tag{11}$$

with λ_j being complex in general and $h(x)$ a regular function. Then, it is immediate that $\int_{a+\epsilon}^{b} \mathrm{d}x\, g(x) = P(1/\epsilon) + H(\epsilon)$, P being a polynomial and $H(0)$ finite. If the $\lambda_j \notin \mathbb{N}$, then one defines the Hadamard regularized integral as

$$⨎_{a}^{b} \mathrm{d}x\, g(x) := \int_{a}^{b} h(x)\, \mathrm{d}x - \sum_{j=1}^{k} \frac{a_j}{\lambda_j - 1}(b-a)^{1-\lambda_j}. \tag{12}$$

Alternatively, one may define, for $\alpha \notin \mathbb{N}$, $p < \alpha < p+1$, and $f^{(p+1)} \in C_{[-1,1]}$, $K^\alpha f := \frac{1}{\Gamma(-\alpha)} \displaystyle\fint_{-1}^{1} \mathrm{d}t \, \frac{f(t)}{(1-t)^{\alpha+1}}$, to obtain, after some steps,

$$K^\alpha f = \sum_{j=0}^{p} \frac{f^{(j)}(-1)}{\Gamma(j+1-\alpha)2^{\alpha-j}} + \frac{1}{\Gamma(p+1-\alpha)} - \int_{-1}^{1} (1-t)^{p-\alpha} f^{(p+1)}(t) \tag{13}$$

where the last integral is at worst improper (Cauchy's principal part). If $\lambda_1 = 1$, the result is $a_1 \ln(b-a)$, instead. If $\lambda_1 = p \in \mathbb{N}$, calling $H_p(f;x) := \fint_{-1}^{1} \mathrm{d}t \, \frac{f(t)}{(t-x)^{p+1}}$, $|x| < 1$, we get

$$H_p(f;x) = \int_{-1}^{1} \left[f(t) - \sum_{j=0}^{p} \frac{f^{(j)}(x)}{j!}(t-x)^j \right] \frac{\mathrm{d}t}{(t-x)^{p+1}} + \frac{f^{(j)}(x)}{j!} \fint_{-1}^{1} \frac{\mathrm{d}t}{(t-x)^{p+1-j}} \tag{14}$$

where the first term is regular and the second one can be easily reduced to

$$\frac{1}{(p-j)!} \frac{\mathrm{d}^{p-j}}{\mathrm{d}x^{p-j}} \fint_{-1}^{1} \frac{\mathrm{d}t}{t-x} \tag{15}$$

being the last integral, as before, a Cauchy PP.

An alternative form of Hadamard's regularization, which is more fashionable for physical applications (as is apparent from the expression itself) is the following [12]. For the case of two singularities, at $\vec{x}_1, \vec{x}_2$, after excising from space two little balls around them, $\mathbb{R}^3 \setminus \left(B_{r_1}(\vec{x}_1) \cup B_{r_2}(\vec{x}_2) \right)$, with $B_{r_1}(\vec{x}_1) \cap B_{r_2}(\vec{x}_2) = \emptyset$, one defines the regularized integral as being the finite part of the limit

$$\fint \mathrm{d}^3x \, F(\vec{x}) := \mathrm{FP}_{\alpha,\beta\to 0} \int \mathrm{d}^3x \left(\frac{r_1}{s_1} \right)^\alpha \left(\frac{r_2}{s_2} \right)^\beta F(\vec{x}) \tag{16}$$

where s_1 and s_2 are two (dimensionful) regularization parameters [12]. This is the version that will be employed in what follows.

5. Hadamard regularization of the Casimir effect

We now use Hadamard's regularization as an additional tool in order to make sense of the infinite expressions encountered in the boundary value problems considered before. As it turns out from a detailed analysis of the results in [7] (which we shall not repeat here, for conciseness), the basic integrals which produce infinities, in the one-dimensional and two-dimensional cases there considered, are the following.

In one dimension, with Dirichlet BC imposed at one ($x = 0$) and two ($x = \pm a$) points, respectively, by means of a delta-background of strength λ (see [7]), one encounters the two divergent integrals:

$$E_1(\lambda, m) = \frac{1}{2\pi} \int_m^\infty \frac{\mathrm{d}t}{\sqrt{t^2 - m^2}} \left[t \log\left(1 + \frac{\lambda}{2t}\right) - \frac{\lambda}{2} \right] \tag{17}$$

$$E_2(a, \lambda, m) = \frac{1}{2\pi} \int_m^\infty \frac{\mathrm{d}t}{\sqrt{t^2 - m^2}} \left\{ t \log\left[1 + \frac{\lambda}{t} + \frac{\lambda^2}{4t^2}(1 - \mathrm{e}^{-4at}) \right] - \lambda \right\}. \tag{18}$$

Using Hadamard's regularization, as described before, we obtain for the first one, equation (17),

$$E_1(m) = \frac{\lambda}{4\pi} \left(1 - \ln \frac{\lambda}{m} \right) \Bigg|_{\lambda\to\infty} + \fint \tag{19}$$

where the first term is the singular part when the limit $\lambda \to \infty$ is taken, and the second—which is Hadamard's finite part—yields in this case

$$\fint = -\frac{m}{4}. \tag{20}$$

Such result is coinciding with the classical one (0, for $m = 0$). Note in particular, that the further $\ln m$ divergence as $m \to \infty$ is hidden in the λ-divergent part, and this behaviour does explain why the classical results which are obtained using hard Dirichlet BC—which corresponds as we just prove here to the Hadamard's regularized part—cannot see it.

In the case of a two-point boundary at $x = \pm a$ (separation $2a$), equation (18), we get a similar equation (19) but now the regularized integral is as follows. For the massless case, we obtain

$$\fint = -\frac{\pi}{48a} \tag{21}$$

which is the regularized result to be found in the classical books. In the massive case, $m \neq 0$, after some additional work the following fast convergent series turns up (cf equation (10))

$$\fint = -\frac{m}{2\pi}\sum_{k=1}^{\infty}\frac{1}{k}K_1(4akm). \tag{22}$$

Thus equation (19) yields strictly the same result (10) that was already obtained by imposing the Dirichlet BC *ab initio*. What has now been *gained* is a more clear identification of the singular part, in terms of the strength of the delta potential at the boundary. This will be the general conclusion, common to all the other cases considered here.

Correspondingly, for the Casimir force we obtain the finite values[6]

$$F_2(a) = -\frac{\pi}{96a^2} \tag{23}$$

in the massless case, and in the massive one

$$F_2(a, m) = -\frac{m^2}{\pi}\sum_{k=1}^{\infty}\left[K_0(4akm) + \frac{1}{4akm}K_1(4akm)\right]. \tag{24}$$

These expressions coincide with those derived in the above-mentioned textbooks on the Casimir effect, and reproduced before by using the zeta-function method (just take minus the derivative of equation (19) w.r.t. $2a$).

The two-dimensional case turns out to be more singular [7]—in part just for dimensional reasons—and requires additional wishful thinking in order to deal with the circular delta function sitting on the circumference where the Dirichlet BC is imposed. Here one encounters the basic singular integral, for the term contributing to the second Born approximation (we use the same notation as in [7]),

$$\tilde{\sigma}(p) = \int_0^{\infty} \mathrm{d}r\, r\, J_0(pr)\sigma(r) \qquad \sigma(r) = b\lambda \exp\left[-\frac{(r-a)^2}{2\omega^2}\right] \tag{25}$$

with J_0 being a Bessel function of the first kind, and $\int_0^{\infty} \mathrm{d}r\, \sigma(r) = \lambda$, $\sigma(r) \xrightarrow{\omega\to 0} \lambda\delta(r-a)$. Hadamard's regularization yields now (the τ replacing the σ in the regularized version)

$$\tau(r, p) = c\lambda(rp+1)^{-\omega/2} \exp\left[-\frac{(r-a)^2}{2\omega^2}\right] \xrightarrow{\omega\to 0} \lambda\delta(r-a) \tag{26}$$

[6] Note that the force $F(a)$ is given here as minus the derivative of the total energy $E(a)$ w.r.t. $2a$, since this is the distance between the two Dirichlet points (*not* a).

with p being a (dimensionful) regularization parameter, the constant c being given by $c^{-1} = \int_0^\infty \mathrm{d}r\, r^{-\omega} \exp\left[-\frac{(r-a)^2}{2\omega^2}\right]$, which exists and is perfectly finite; in particular, $c^{-1}(\omega = 0.1, a = 1) = 0.25$. Then,

$$\tilde{\tau}(p) = 2\pi \int_0^\infty \mathrm{d}r\, r\, J_0(pr)\tau(r, p) = 2\pi\lambda a(ap+1)^{-\omega/2} J_0(ap). \tag{27}$$

It turns out that, for the Casimir energy, we get in this case (notation as in [7])

$$\begin{aligned} E^{(2)}_{\lambda^2}[\tau] &= \frac{\lambda^2 a^2}{8} \int_0^\infty \mathrm{d}p\, (ap+1)^{-\omega} J_0(ap)^2 \arctan(p/2m)|_{\omega\to 0} \\ &= \frac{\lambda^2 a^2}{8} \left\{ \frac{1}{2\omega} + \frac{\gamma + 3\ln 2}{2a} + 4m \left[\gamma - \frac{2}{\sqrt{\pi}} [1 - \ln(am)]\, h(4a^2m^2) \right] \right\} \end{aligned} \tag{28}$$

where $h(z) := {}_2F_3\,((1/2, 1/2); (1, 1, 3/2); z)$ and γ is the Euler–Mascheroni constant; in particular, for instance $h(1) = 1.186\,711$, which is quite a nice value. Recall also that ω is the *width* of the Gaussian δ, which is the very *physical* parameter considered in [5]. When this width tends to zero an infinite energy appears (the width controls the formation of the pole). The rest of the result is the Hadamard regularization of the integral, e.g.[7]

$$\neq\!\!\!\!\!\!\int_0^\infty \mathrm{d}p\, J_0(ap)^2 \arctan(p/2m). \tag{29}$$

Again, the finite part reverts to the results obtained in the literature with Dirichlet BC *ab initio*.

To summarize, it has been proved here—in some particular but rather non-trivial and representative examples—that the finite results derived through the use of Hadamard's regularization exactly coincide with the values obtained using the more classical, less fully-fledged methods to be found in the literature on the Casimir effect. Moreover, Hadamard's prescription is able to separate and identify the singularities as physically meaningful cut-offs. Although the validity of this additional regularization is at present questionable, the fact that it bridges the two approaches is already remarkable, maybe again a manifestation of the unreasonable effectiveness of mathematics.

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[7] It should be pointed out that the computational program Mathematica [17] directly assigns the Hadamard regularized value to particular cases of integrals of this kind; but it does so without any hint on what is going on. This has often confused many users, who fail to understand how it comes that an infinite integral gets a finite value out of nothing.

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